Actions on tensor products and Kashiwara's crystals

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April 23, 2025

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Continuous symmetries correspond to Lie groups, such as

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Lie groups admit "infinitesimal versions", called Lie algebras.
Corresponding to $SL(2)$, we have the Lie algebra $\mathfrak{sl}_2$ spanned by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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More formally, it is the vector space spanned by $\{h, e, f\}$ with relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

We recover $SL(2)$ by the exponential map e^{tx} with $x \in \mathfrak{sl}_2$.

Let A be an algebra (a vector space with a bilinear product). An action (or representation) of A on a vector space V is a linear map

$$\pi : A \to V \quad \text{such that} \quad \pi(ab) = \pi(a)\pi(b).$$

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Such a map gives a representation on $V \otimes W$ as follows.
Writing $\Delta(a) = \sum_i a_i \otimes a_i'$ for a given $a \in A$, we set

$$\pi_{V\otimes W}(a) = \sum_i \pi_V(a_i) \otimes \pi_W(a_i').$$

For a group, the tensor product representation is given by

$$\pi_{V\otimes W}(g)(v\otimes w) = \pi_V(g)v \otimes \pi_W(g)w.$$

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Similarly, for a Lie algebra we consider

$$\pi_{V\otimes W}(x)(v \otimes w) = \pi_V(x)v \otimes w + v \otimes \pi_W(x)w.$$

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(One way to think about the second example is to take $g=e^{tx}$ in the first example and retain only the terms linear in t).

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We will discuss some other interesting examples shortly.

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A representation V of A is called irreducible (or simple) if the only invariant subspaces are $W = V$ and $W = \{0\}$.

For instance, consider the vector space $V_{1/2}$ with basis $\{v_+, v_-\}$. We have an irreducible representation of $\mathfrak{sl}_2$ on $V_{1/2}$ given by

$$\pi(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(This is the so-called fundamental representation of $\mathfrak{sl}_2$).

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is also invariant under $\mathfrak{sl}_2$. Hence we have a decomposition

$$V_{1/2} \otimes V_{1/2} = V_1 \oplus V_0$$

in terms of simple (irreducible) components.

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In physics terminology, the identification between elements of $V \otimes W$ and elements of some V_i is given by the Clebsch–Gordan coefficients.

Now let us consider the universal enveloping algebra $U(\mathfrak{sl}_2)$.
It is the algebra generated by the elements $\{h, e, f\}$ with relations

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We can define a deformation of $U(\mathfrak{sl}_2)$ as follows. Fix $q \in \mathbb{R}$ and consider the algebra generated by $\{H, E, F\}$ with relations

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This formally recovers the algebra $U(\mathfrak{sl}_2)$ in the limit $q \to 1$, since

$$\frac{q^H - q^{-H}}{q - q^{-1}} = H + O(q - 1).$$

It is convenient to define $K = q^H$ and rewrite the relations as

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

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There is however a more complicated coproduct which works, given by

$$\begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H, \\ \Delta(E) &= E \otimes K^{-1} + 1 \otimes E, \\ \Delta(F) &= F \otimes 1 + K \otimes F. \end{aligned}$$

(We note that the first relation implies $\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}$).

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Using these expressions we compute

$$\begin{aligned}
[\Delta(E),\Delta(F)] &= [E,F]\otimes K^{-1} + K\otimes[E,F] \\
&= \frac{1}{q-q^{-1}}\left((K-K^{-1})\otimes K^{-1} + K\otimes(K-K^{-1})\right) \\
&= \frac{1}{q-q^{-1}}(K\otimes K - K^{-1}\otimes K^{-1}) \\
&= \Delta\left(\frac{K-K^{-1}}{q-q^{-1}}\right).
\end{aligned}$$

(We also used the identity $EK\otimes K^{-1}F = KE\otimes FK^{-1}$).

It turns out that, by a similar procedure, one can define a quantized enveloping algebra $U_q(\mathfrak{g})$ for any (complex) semisimple Lie algebra $\mathfrak{g}$.

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The generators are $\{H_i, E_i, F_i\}_{i=1}^r$, where r is the rank of $\mathfrak{g}$, but the relations among them are more complicated. Its coproduct is

$$\begin{aligned}\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i,\\ \Delta(E_i) &= E_i \otimes K_i^{-1} + 1 \otimes E_i,\\ \Delta(F_i) &= F_i \otimes 1 + K_i \otimes F_i.\end{aligned}$$

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There are many similarities between $U_q(\mathfrak{g})$ and the classical $U(\mathfrak{g})$. For instance, their irreducible representations can be put in one-to-one correspondence (and in turn with those of the Lie algebra $\mathfrak{g}$).

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For instance consider the fundamental representation $V^q_{1/2}$, given by

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The action on the tensor product $V^q_{1/2} \otimes V^q_{1/2}$ can be obtained readily. Using the rule $\Delta(F) = F \otimes 1 + K \otimes F$ we obtain

$$\begin{aligned} F(v_+ \otimes v_+) &= v_- \otimes v_+ + qv_+ \otimes v_-, \\ F(v_+ \otimes v_-) &= v_- \otimes v_-, \\ F(v_- \otimes v_+) &= qv_- \otimes v_-, \\ F(v_- \otimes v_-) &= 0. \end{aligned}$$

We have a decomposition $V^q_{1/2} \otimes V^q_{1/2} = V^q_1 \oplus V^q_0$ into simple components analogous to the classical case, where

$$V^q_1 = \mathrm{span}\{v_+ \otimes v_+,\ v_- \otimes v_+ + qv_+ \otimes v_-,\ v_- \otimes v_-\},$$
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Something intriguing can be seen here: for $q = 0$ we get

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We can interpret this as follows: our "naive basis" $\{v_i \otimes v_j\}$ of $V^q_{1/2} \otimes V^q_{1/2}$ gets partitioned into irreducible components.

This can be represented graphically in the so-called crystal graph:

$$\begin{tikzcd} v_+ \otimes v_+ \arrow[r] & v_- \otimes v_+ \arrow[d] \\ v_+ \otimes v_- & v_- \otimes v_- \end{tikzcd}$$

Its vertices are simply the elements of the basis $\{v_i \otimes v_j\}$.
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And what about $U_q(\mathfrak{g})$ for a general semisimple Lie algebra $\mathfrak{g}$?

The answer is yes for both, due to the results of Kashiwara!
The (appropriate) limit $q \to 0$ of $U_q(\mathfrak{g})$ is called the crystal limit.

A crystal $\mathcal{B}$ is a set on which the Kashiwara operators $\tilde{E}_i$ and $\tilde{F}_i$ act.
(These are renormalizations of the actions of E_i and F_i at $q = 0$).
It should satisfy certain properties that we will not list here.

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One of the main features of crystals is the tensor product rule. To formulate it, we need to introduce the following numbers:

$$\varphi_i(b) = \max\{n : \tilde{F}_i^n b \neq 0\}, \quad \varepsilon_i(b) = \max\{n : \tilde{E}_i^n b \neq 0\}.$$

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Theorem (Kashiwara '91)

The action of $\tilde{F}_i$ on (the Cartesian product) $\mathcal{B} \times \mathcal{B}'$ is given by

$$\tilde{F}_i(b \otimes b') = \begin{cases} \tilde{F}_i b \otimes b' & \varphi_i(b) > \varepsilon_i(b') \\ b \otimes \tilde{F}_i b' & \varphi_i(b) \leq \varepsilon_i(b') \end{cases}.$$

Compare with the action of F_i on the tensor product $V \otimes V'$ of $\mathfrak{g}$-representations, which would be $F_i(v \otimes v') = F_i v \otimes v' + v \otimes F_i v'$.

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For instance, in the example of $\mathfrak{sl}_2$, we saw the crystal graph

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Theorem (Kashiwara '91)

The decomposition of $\mathcal{B} = \bigsqcup_k \mathcal{B}_k$ into connected components corresponds to the decomposition $V = \bigoplus_k V_k$ into simple components.

Let us see a more complicated example of this. Consider two copies of the three-dimensional representation V_1 of $\mathfrak{sl}_2$, which we denote by

$$a_1 \to a_2 \to a_3, \quad b_1 \to b_2 \to b_3.$$

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Then the crystal graph corresponding to $V_1 \otimes V_1$ is given by

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We get three connected components, of cardinality 5, 3 and 1. This corresponds to the $\mathfrak{sl}_2$-decomposition

$$V_1 \otimes V_1 = V_2 \otimes V_1 \otimes V_0.$$

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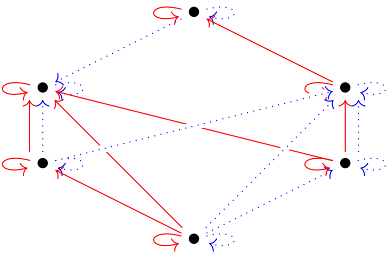

But this is a story for another time...

The two main papers by Kashiwara on crystal bases.

- M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Communications in Mathematical Physics 133, no. 2 (1990): 249-260.
- M. Kashiwara, *On crystal bases of the Q-analogue of universal enveloping algebras*, Duke Mathematical Journal 63, no. 2 (1991): 465-516.

Some (more user-friendly) textbook presentations of crystal bases.

- D. Bump and A. Schilling, *Crystal bases: Representations and Combinatorics*, World Scientific Publishing Company, 2017.
- J. Hong and S. Kang, *Introduction to quantum groups and crystal bases*, American Mathematical Society 42, 2002.

Finally, the reference for the last briefly mentioned topic.

- M. Matassa and R. Yuncken, *Crystal limits of compact semisimple quantum groups as higher-rank graph algebras*, Journal für die reine und angewandte Mathematik (Crelles Journal) 802 (2023): 173-221.