A rigorous formulation of density functional theory for spinless electrons in one dimension

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Overview

Introduction

Motivation: the ground-state problem in QM

Quantum Mechanics: Let $\Omega \subset \mathbb{R}^d$ be a physical space, and consider the Schrödinger operator

$$H_N(v, w) = -\Delta + \sum_{i<j}^{N} w(x_i, x_j) + \sum_{j=1}^{N} v(x_j),$$

acting on the N-particle (spinless) fermionic space

$$\mathcal{H}_N(\Omega) := \wedge^N \mathrm{L}^2(\Omega; \mathbb{C})$$

Here, v and w and the external and interaction potential (given by physics of the system).

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Ground-state problem: find minimum energy and minimizer of

$$\mathcal{E}_{\mathrm{GS}}(v, w) = \min_{\Psi \in \mathcal{H}_N} \frac{\langle \Psi, H_N(v, w)\Psi \rangle}{\|\Psi\|_{\mathrm{L}^2}^2}.$$

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Curse of dimension: exponential cost in N!

Density Functional Theory

DFT idea: shift the focus to ground-state density

$$\rho_\Psi(x) = N \int_{\Omega^{N-1}} |\Psi(x, x_2, ..., x_N)|^2 \mathrm{d}x_2 ... \mathrm{d}x_N.$$

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How?

Step 1: the Levy-Lieb constrained search

$$F_{\mathrm{LL}}(\rho; w) = \inf_{\Psi \mapsto \rho} \langle \Psi, H_N(0, w)\Psi \rangle,$$

where ρ belongs to the set of N-representable densities

$$\mathcal{R}_N := \left\{ \rho : \sqrt{\rho} \in \mathrm{H}^1(\Omega), \quad \int_\Omega \rho(x)\mathrm{d}x = N, \quad \rho \geq 0 \right\}$$

Then

$$\mathcal{E}_{\mathrm{GS}}(v, w) = \inf_{\rho \in \mathcal{R}_N} \{F_{\mathrm{LL}}(\rho; w) + v(\rho)\}$$

The Kohn-Sham Scheme

Step 2: Introduce the exchange-correlation functional

$$E_{\mathrm{xc}}(\rho; w) := F_{\mathrm{LL}}(\rho; w) - E_H(\rho; w) - T_{\mathrm{KS}}(\rho),$$

where E_H is the Hartree energy

$$E_H(\rho; w) = w(\rho \otimes \rho) = \int_{\Omega^2} \rho(x) w(x, y) \rho(y) \mathrm{d}x \mathrm{d}y$$

and T_{KS} is the Kohn-Sham kinetic energy

$$T_{\mathrm{KS}}(\rho) = \inf_{\substack{\Psi \in \mathcal{S}_N \\ \Psi \mapsto \rho}} \int_{\Omega^N} |\nabla \Psi|^2 (x_1, ..., x_N) \mathrm{d}x_1 ... \mathrm{d}x_N,$$

and $\mathcal{S}_N = \{\varphi_1 \wedge ... \wedge \varphi_N : \langle \varphi_i, \varphi_j \rangle = \delta_{ij}\}$ is the set of normalized Slater determinants. Then, clearly,

$$\mathcal{E}_{\mathrm{GS}}(v, w) = \inf_{\rho \in \mathcal{R}_N} \{T_{\mathrm{KS}}(\rho) + E_H(\rho; w) + E_{\mathrm{xc}}(\rho; w) + v(\rho)\}$$

The Kohn-Sham equations

Step 3: Reformulate over orbitals and differentiate

$$\mathcal{E}_{\mathrm{GS}}(v, w) = \inf_{\Phi \in \mathcal{S}_N} \{\|\nabla \Phi\|_{\mathrm{L}^2}^2 + E_H(\rho_\Phi; w) + E_{\mathrm{xc}}(\rho_\Phi; w) + v(\rho_\Phi)\}$$

Computing the Euler-Lagrange equation for orbitals under orthogonality constraint leads to the Kohn-Sham equations

$$(-\Delta + v_{\mathrm{xc}}(\rho_\Phi) + v_H(\rho_\Phi) + v)\,\varphi_j = \lambda_j \varphi_j,$$

where φ_j are the orbitals of the minimizer (or critical point) $\Phi \in \mathcal{S}_N$ and $v_H(\rho_\Phi)$ is the Hartree potential

$$v_H(\rho) = \int_\Omega w(x, y)\rho_\Phi(y)\mathrm{d}y,$$

and $v_{\mathrm{xc}}(\rho_\Phi)$ is the xc-potential

$$v_{\mathrm{xc}}(\rho_\Phi) = \mathrm{d}_\rho E_{\mathrm{xc}}(\rho_\Phi).$$

Remark: $v_{\mathrm{xc}}(\rho)$ is the Gateaux derivative (whatever this means) of E_{xc} at ρ.

Exact KS-DFT (?)

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(2) (Existence) Let ρ be the density of a ground-state of $H_N(v, w)$, then is there a potential $v_{\mathrm{KS}}(\rho)$ such that $\rho = \rho_\Phi = \sum_{j=1}^{N} |\varphi_j|^2$ for some eigenfunctions φ_j of the single-particle (Kohn-Sham) operator $h_{\mathrm{KS}}(\rho) = -\Delta + v_{\mathrm{KS}}(\rho)$?

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Some Remarks:

$(*)$ If (1) holds, then existence of (at least) one potential is guaranteed, namely

$$v_{\mathrm{KS}}(\rho) = v_{\mathrm{H}}(\rho) + v_{\mathrm{xc}}(\rho) + v.$$

$(*)$ Question (2) is called the v-representability question and φ_j are expected to be the lowest eigenfunctions of h_{KS} (Aufbau principle).

$(*)$ Question (3) is related to the Hohenberg-Kohn (HK) theorem.

The 1D case: a light at the end of the tunnel

For the rest of this talk, we focus on *one-dimensional* systems, i.e., $\Omega = I = (0,1) \subset \mathbb{R}$ or $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$.

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Recent work by Sutter, Penz, Ruggenthaler, Leeuwen, and Giesbertz, they consider $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ (1D torus) and take distributional potentials

$$\boxed{\mathcal{V}(\mathbb{T}) = \mathrm{H}^{-1}(\mathbb{T}) = \{v : \mathrm{H}^1(\mathbb{T}) \to \mathbb{C} \quad \text{linear continuous and real}\}},$$

where $\mathrm{H}^1(\mathbb{T})$ is the Sobolev space of L^2 (periodic) functions with L^2 integrable gradient.

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Theorem (Sutter et al (2024))

For any $N \in \mathbb{N}$, w in a large class of interactions and ρ in the set

$$\mathcal{D}_N := \left\{\rho : \sqrt{\rho} \in \mathrm{H}^1(\mathbb{T}), \quad \int_{\mathbb{T}} \rho = N, \quad \rho > 0\right\}$$

there exists $v \in \mathcal{V}(\mathbb{T})$ such that ρ is a convex combination of ground-state densities of $H_N(v, w)$.

Questions left open in 1D

Goal of this talk: Answer the previous question in the case of 1D systems.

(a) Are there more ensemble $\mathcal{V}(\mathbb{T})$-representable densities?

(b) What about pure-state $\mathcal{V}(\mathbb{T})$-representability?

(c) Are the potentials representing $\rho \in \mathcal{D}_N$ unique?

(d) Is the xc-functional E_{xc} differentiable, is the Kohn-Sham scheme *rigorously* exact?

Results

Results: the setting

We set $I = (0, 1)$ and define the spaces of distributional external and interaction potentials:

$$\boxed{\mathcal{V}(I) := \mathrm{H}^{-1}(I; \mathbb{R})} \qquad \text{and} \qquad \mathcal{W}(I) := \{w \in \mathrm{W}^{-1,p}(I \times I; \mathbb{R}) : p > 2\}$$

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Then, it is fairly simple to show that there exists a unique self-adjoint operator

$$H_N(v,w) = -\Delta + \sum_{i \neq j}^{N} w(x_i, x_j) + \sum_{j=1}^{N} v(x_i)$$

associated to the quadratic form

$$\mathfrak{a}_{v,w}(\Psi,\Psi) = \int_{I^N} |\nabla \Psi|^2 \mathrm{d}x + v(\rho_\Psi) + w(\rho_\Psi^{(2)}),$$

with form domain $Q_N = \mathrm{H}^1(I^N) \cap \mathcal{H}_N(I)$. Here ρ_Ψ is the density and $\rho_\Psi^{(2)}$ is the pair density

$$\rho_\Psi^{(2)}(x,y) := N(N-1) \int_{I^{N-2}} |\Psi(x,y,x_3,...,x_N)|^2 \mathrm{d}x_3 ... \mathrm{d}x_N.$$

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$$\rho_\Psi^{(2)}(x,y) := N(N-1) \int_{I^{N-2}} |\Psi(x,y,x_3,...,x_N)|^2 \mathrm{d}x_3 ... \mathrm{d}x_N.$$

Remarks: (1) As $\mathrm{H}^1(I^N)$ is compactly embedded in $\mathrm{L}^2(I^N)$, the spectrum of $H_N(v,w)$ is discrete.
(2) The form domain considered here is also called the Neumann form domain.

Results: the pure-state v-representability problem

In this setting, one can obtain a complete solution of the v-representability problem.

Theorem (Characterization of pure-state $\mathcal{V}$-representability)

Let $w \in \mathcal{W}(I)$ be fixed and $N \in \mathbb{N}$. Then a function ρ is the density of a ground-state of $H_N(v, w)$ for some $v \in \mathcal{V}(I)$ if and only if ρ belongs to the set

$$\mathcal{D}_N(w) = \left\{ \rho : \sqrt{\rho} \in \mathrm{H}^1(I), \quad \int_I \rho(x)\mathrm{d}x = N \quad \text{and} \quad \rho(x) > 0 \quad \text{for any } x \in [0,1] \right\}. \qquad (1)$$

In particular, $\mathcal{D}_N(w) = \mathcal{D}_N$ is independent of the interaction potential $w \in \mathcal{W}(I)$.

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In particular, $\mathcal{D}_N(w) = \mathcal{D}_N$ is independent of the interaction potential $w \in \mathcal{W}(I)$.

Remark: In the case of periodic and anti-periodic BCs, one can show that $\mathcal{D}^{\mathrm{Per}}_{2k-1}(w) = \mathcal{D}^{+}_{2k-1}$ and $\mathcal{D}^{\mathrm{Ant}}_{2k}(w) = \mathcal{D}^{+}_{2k}$ for any $k \in \mathbb{N}$, where

$$\mathcal{D}^{+}_N := \left\{ \rho \in \mathrm{H}^1(I) : \int_I \rho(x)\mathrm{d}x = N, \quad \rho(0) = \rho(1), \quad \text{and} \quad \rho(x) > 0 \quad \text{for any } x \in I \right\}.$$

Results: Hohenberg-Kohn theorem

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Answer: no!

Theorem (Hohenberg-Kohn Theorem)

Let $w \in \mathcal{W}(I)$, $N \in \mathbb{N}$, and $v, v' \in \mathcal{V}(I)$ be such that $H_N(v, w)$ and $H_N(v', w)$ have the same ground-state density. Then there exists a constant $c > 0$ such that

$$v - v' = c.$$

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Remark: The previous two results show that, for any $w \in \mathcal{W}(I)$, there exists a bijective map

$$\rho \in \mathcal{D}_N \mapsto v(\rho; w) \in \mathcal{V}(I)/\{1\},$$

where $\mathcal{V}(I)/\{1\}$ is the space of equivalence classes in $\mathcal{V}(I)$ modulo additive constants.

Results: differentiability of the xc-functional

By combining the previous results, we can show that E_{xc} is differentiable!

Theorem (Differentiability of the xc-functional)

Recall that

$$E_{\mathrm{xc}}(\rho) = F_{\mathrm{LL}}(\rho; w) - E_H(\rho; w) - T_{\mathrm{KS}}(\rho).$$

Then for any $\rho \in \mathcal{D}_N$, the map $\rho \in \mathcal{R}_N \mapsto E_{\mathrm{xc}}(\rho)$ is Gateaux differentiable at ρ. In other words, there exists (a unique) $v_{\mathrm{xc}}(\rho) \in \mathcal{V}(I)/\{1\}$ such that

$$\lim_{\epsilon \downarrow 0} \frac{E_{\mathrm{xc}}(\rho + \epsilon\delta) - E_{\mathrm{xc}}(\rho)}{\epsilon} = v_{\mathrm{xc}}(\rho)(\delta), \quad \text{for any} \quad \delta \in \mathcal{T}_\rho \mathcal{D}_N := \{\delta \in \mathrm{H}^1(I; \mathbb{R}), \quad \int_I \delta(x)\mathrm{d}x = 0\}.$$

Results: the KS-scheme is rigorously exact

Corollary (Exact Kohn-Sham DFT)

Let $v \in \mathcal{V}$, $w \in \mathcal{W}$ and $N \in \mathbb{N}$, and denote by $\rho(v;w) \in \mathcal{D}_N$ the (unique) ground-state density of $H_N(v,w)$. Then, there exists a unique (up to a global phase) minimizer of

$$\min_{\substack{\Psi \in \mathcal{S}_N \\ \|\Psi\|=1}} \{T_{\mathrm{KS}}(\rho_\Psi) + E_{\mathrm{xc}}(\rho_\Psi) + E_H(\rho_\Psi) + v(\rho_\Psi)\}.$$

Moreover, this minimizer is given by the Slater determinant of the N lowest eigenfunctions of the Kohn-Sham single-particle Hamiltonian

$$h(\rho(v;w)) = -\Delta + v_{\mathrm{xc}}(\rho(v;w)) + v_H(\rho(v;w)) + v,$$

where $v_{\mathrm{xc}}(\rho) = \mathrm{d}_\rho E_{\mathrm{xc}} \in \mathcal{V}$ and $v_H(\rho) \in \mathcal{V}$ is the Hartree (distributional) potential

$$\delta \mapsto v_H(\rho)(\delta) = w(\rho \otimes \delta) + w(\delta \otimes \rho). \tag{2}$$

On the proofs

A non-degeneracy theorem for spinless electrons in 1D

A key ingredient in the proof of the previous results is the following non-degeneracy theorem.

Theorem (Non-degeneracy theorem for spinless electrons in 1D)

Let $v \in \mathcal{V}(I)$, $w \in \mathcal{W}(I)$, and $N \in \mathbb{N}$. Then the ground-state Ψ of $H_N(v, w)$ is non-degenerate and we have $|\{\Psi = 0\}| = 0$.

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Proof: Need two main steps. **1st Step:** A maximum principle (or Perron-Frobenius theorem) for Schrödinger operators with distributional potentials.

Theorem (Perron-Frobenius theorem)

Let $\Omega \subset \mathbb{R}^d$ be open bounded and Lipschitz, let $X \subset \mathrm{H}^1(\Omega)$ be a closed subspace such that $C_c^\infty(\Omega) \subset X$ and $f \in X$ implies $|f| \in X$. Then the self-adjoint realization of $-\Delta_X + V$ for suitable (distributional) potentials with form domain X has a non-degenerate ground-state and the ground-state eigenfunction is (up to a global phase) almost everywhere strictly positive in Ω.

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Problem: For $X = \mathrm{H}^1(I^N) \cap \mathcal{H}_N$, it is not true that $|f| \in X$ for any $f \in X$.

Domain slicing or reduction to the simplex

It turns out that if Ψ is anti-symmetric with respect to each 1D coordinate, then it is fully determined by its values on the simplex

$$S_N = \{(x_1, ..., x_N) \in [0,1]^N : 0 < x_1 < x_2 < x_3 .... < x_N < 1\}$$

In other words, we can split the domain and transform the operator $H_N(v, w)$ into a (slightly different) Schrödinger operator acting on $\mathrm{L}^2(S_N)$. In particular, reduced operator has form domain

$$\mathrm{H}^1_\Gamma(S_N) = \{\Psi \in \mathrm{H}^1(S_N) : \Psi = 0 \quad \text{along } \Gamma_{\mathrm{int}}.\}$$ No particle symmetry!

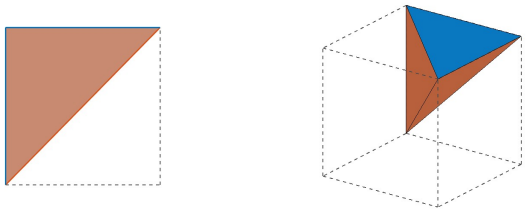

Figure: Simplex in the case of $N = 2$ and $N = 3$ particles.

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where the infimum runs over the set of mixed-state density matrices. Then $F_{\mathrm{L}}(\rho)$ is convex and $\rho \in \mathcal{R}_N$ is ensemble $\mathcal{V}$-representable if and only if $\partial F_{\mathrm{L}}(\rho) \neq \emptyset$.

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2nd Step: Apply convex analysis argument to show that any $\rho \in \mathcal{D}_N$ is ensemble $\mathcal{V}$-rep. This relies on the fact

$$\mathrm{ri}(\mathcal{R}_N) = \mathcal{D}_N = \{\rho \in \mathcal{R}_N : \rho(x) > 0 \quad \text{for any } x \in [0,1]\}.$$

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3rd Step: by the non-degeneracy theorem ensemble $\mathcal{V}$-rep is equal to pure-state $\mathcal{V}$-rep.

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To obtain sufficient conditions for pure-state v-rep, we adapt the argument of Sutter et al and use the non-degeneracy theorem.

1st Step: Introduced the Lieb convex functional

$$F_{\mathrm{L}}(\rho) = \inf_{\Pi \mapsto \rho} \{\mathrm{Tr}\, \Pi\, H_N(0,w)\}, \quad \rho \in \mathcal{R}_N = \{\rho : \sqrt{\rho} \in \mathrm{H}^1, \int \rho = N, \rho \geq 0\},$$

where the infimum runs over the set of mixed-state density matrices. Then $F_{\mathrm{L}}(\rho)$ is convex and $\rho \in \mathcal{R}_N$ is ensemble $\mathcal{V}$-representable if and only if $\partial F_{\mathrm{L}}(\rho) \neq \emptyset$.

2nd Step: Apply convex analysis argument to show that any $\rho \in \mathcal{D}_N$ is ensemble $\mathcal{V}$-rep. This relies on the fact

$$\mathrm{ri}(\mathcal{R}_N) = \mathcal{D}_N = \{\rho \in \mathcal{R}_N : \rho(x) > 0 \quad \text{for any } x \in [0,1]\}.$$

3rd Step: by the non-degeneracy theorem ensemble $\mathcal{V}$-rep is equal to pure-state $\mathcal{V}$-rep.

Conclusion: any $\rho \in \mathcal{D}_N$ is pure-state $\mathcal{V}$-rep and $F_{\mathrm{LL}}(\rho) = F_{\mathrm{L}}(\rho)$ for any $\rho \in \mathcal{D}_N$.

Necessary conditions for v-rep - nodal domain theorem and boundary UCP

To prove that every ground-state density belongs to $\mathcal{D}_N$ we extend some well-known results in the spectral theory of Schrödinger operators to the case of distributional potentials. More precisely, we use:

(i) To show that $\rho_\Psi > 0$ inside the interval $I = (0, 1)$, we use (an generalized version of) Courant's nodal domain theorem and the non-degeneracy theorem

Theorem (Courant's nodal domain theorem)

Let Ψ be the n^{th} eigenfunction of $-\Delta + v$, then Ψ has at most n nodal domains.

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(ii) To show that $\rho_\Psi > 0$ on the end points $x_0 \in \{0, 1\}$, we prove the following weak continuation along the boundary

Theorem (UCP on the boundary)

Let $U \subset \partial[0, 1]^N$ be relatively open, then Ψ can not vanish identically in U.

The HK theorem with distributional potentials

1st Step (Standard HK argument): Let $v, v' \in \mathcal{V}(I)$ be potentials with same ground-state density, then Ψ is mutual ground-state of $H_N(v, w)$ and $H_N(v', w)$. Hence

$$\sum_{j=1}^{N}(v - v')(x_i)\Psi(x_1, ..., x_N) = 0 \tag{3}$$

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(ii) Equation (3) only holds in the distributional sense, i.e.,

$$(v - v')(\rho_{\Psi,\Phi}) = 0 \quad \text{where} \quad \rho_{\Psi,\Phi}(x) = N \int_{I^{N-1}} \Psi(x, x_2, ..., x_N)\Phi(x, x_2, ..., x_N)\mathrm{d}x_2...\mathrm{d}x_N. \tag{4}$$

In particular, unique continuation is not enough!

The HK theorem for distributional potentials via regularity improving

Key idea: test (4) with states of the form

$$\Phi_f(x_1, ..., x_N) = \sum_{j=1}^{N} \frac{f(x_i)}{\rho(x_i)} \Psi(x_1, ..., x_N) \qquad f \in \mathrm{H}^1(I).$$

Using Φ_f, we find

$$(v - v')(\rho_{\Psi,\Phi_f}) = (v - v')(f) - (v - v') \left(\int_I \rho_\Psi^{(2)}(x, y) \rho(y)^{-1} f(y) \mathrm{d}y \right) = 0 \quad \text{for any } f \in \mathrm{H}^1(I).$$

Thus $v - v'$ is an eigenfunction of the adjoint of the operator

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One can then show that K^* is regularity improving, i.e., it sends H^{-1} to L^p for some $p \in [1, \infty)$. Therefore $v - v' \in \mathrm{L}^p(I)$ and we can use the GS unique continuation to complete the proof.

Conclusion

Summary

In this talk we have

(∗) Presented a complete solution of the *pure-state* v-representability problem in one-dimensional interval with (generalized) Neumann BCs..

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(∗) Established the differentiability of the xc-functional and therefore obtained a rigorous derivation of the exact KS-scheme.

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(∗) Established the differentiability of the xc-functional and therefore obtained a rigorous derivation of the exact KS-scheme.

All of these were only possible by using the "correct" space of one-body potentials $\mathcal{V}(I) = \mathrm{H}^{-1}(I;\mathbb{R})$.

In particular, we conclude that distributional potentials are **indispensable** to a rigorous foundation of (KS)-DFT.

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Immediate follow-up questions (work in progress):

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(4) And most importantly, what about higher dimensions? What is the correct class of potentials?

In 1D, one can actually show that if $\mathfrak{a}(\psi, \varphi)$ is a local, real-valued, and infinitesimally bounded with respect to $-\Delta$, then there exists a unique $v \in \mathcal{V}(I)$ such that

$$\mathfrak{a}(\psi, \varphi) = v(\overline{\psi}\varphi) \quad \text{for any } \psi, \varphi \in \mathrm{H}^1(I).$$

This suggests that a similar characterization in higher dimensions could be the key to understand the v-rep question in higher dimensions.

Thanks for the attention!

References:

(∗) T. C. Corso, *v-representability and Hohenberg-Kohn theorem for non-interacting Schrödinger operators with distributional potentials on the one-dimensional torus*, J. Phys. A: Math. Theor. 58 125203

(∗) T. C. Corso, *A non-degeneracy theorem for interacting electrons in one dimension*, arXiv:2503.18440.

(∗) T. C. Corso, *A rigorous formulation of Density Functional Theory for spinless electrons in one dimension*, arXiv:2504.05501

Questions?

Proof of Perron-Frobenius Theorem

Proof for the Laplacian relies on two simple ingredients:

(1) Locality of the form $\mathfrak{a}_0(\Psi, \Psi) = \|\nabla \Psi\|^2_{L^2}$,

$$\mathfrak{a}_0(\Psi, \Phi) = 0 \quad \text{if } \Psi\Phi = 0, \text{ i.e., } \Psi \text{ and } \Phi \text{ have (a.e.) disjoint support.}$$

(2) Strong maximum principle for subsolutions of Laplace's equation, i.e., if $\Psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfies $-\Delta\Psi \geq 0$, then

$$\Psi(x_0) = \inf_{x \in \Omega} \{\Psi(x)\} \quad \text{for some } x_0 \in \Omega \text{ if and only if } \Psi \text{ is constant.}$$

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Let Ω and X be as in the previous theorem. Then $-\Delta_X + V$ has a unique and almost everywhere strictly positive ground-state.

Sketch of proof: 1 Step: Show that H has unique strictly positive ground-state if and only if the semigroup e^{-tH} is positivity improving.
2 Step: Apply Trotter product formula

$$e^{-t(H_0+V)} = \mathrm{s} - \lim_{n\to\infty} \left(e^{-(t/n)H} e^{-(t/n)V}\right)^n.$$

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Conclude from Step 1 and previous theorem.

Applies to many (possibly all) distributional potentials satisfying

$$|V(|\Psi|^2)| \leq a\|\Psi\|^2_{\mathrm{H}^1} + C_a\|\Psi\|^2_{\mathrm{L}^2} \quad \text{for } a < 1.$$

A non-degeneracty theorem for anti-symmetric systems

It turns out that one can show the following:

Theorem

Let v, w satisfies some (very general) regularity assumptions, then the ground-state of the self-adjoint operator $H_N(v,w) = -\Delta + \sum_{j=1}^N v(x_j) + \sum_{i\neq j=1}^N w(x_i,x_j)$ defined via the form

$$\mathfrak{a}_{v,w}(\Psi,\Psi) = \int_{[0,1]^N} |\nabla\Psi|^2 \mathrm{d}x + v(\rho_\Psi) + w(\rho_{2,\Psi})$$

with form domain $\mathrm{H}^1([0,1]^N)$ has a non-degenerate ground-state and $\Psi(x) \neq 0$ for almost every $x \in [0,1]^N$.

Moreover, if one considers the self-adjoint realization of $H_N(v,w)$ with periodic or anti-periodic form domain, then the same results when the number of particles is, respectively, odd and even.

Remark: The fact that $\Psi \neq 0$ a.e. is known as the strong unique continuation principle or property (UCP) and is not known for distributional potentials v and w.